PART TWO. CURRENT ISSUES AND CONTROVERSIES
On the Relation of Topological to Metrical Structure

In his inaugural dissertation, Riemann emphasized the distinction between the topology of space, which, for him, meant its continuity or discreteness, and its geometry.¹ Taking the topology as essential, he convincingly argued that the concept of a continuous space does not, by itself, imply a particular geometry, and that therefore the nature of the geometry of space involves physical considerations. This thesis he expressed by the assertion that metrical relations are not implicit in the concept of a continuum, though they are implicit in the concept of a discretum.²

Professor Grünbaum has argued that the ideas of Riemann’s inaugural dissertation establish a much stronger thesis. He takes Riemann to have established that in a nondenumerable and dense space, the self-congruence of the metric standard is conventional, and to this extent, metrical relations generally are conventional. According to this thesis, this means that they are not real relations which may or may not be discovered to obtain,

¹ Here and elsewhere, I mean metric geometry.
² Cf. [4], pp. 424–425: “The question of the validity of the postulates of geometry in the indefinitely small is involved in the question concerning the ultimate basis of relations of size in space. In connection with this question, which may well be assigned to the philosophy of space, the above remark is applicable, namely that while in a discrete manifold the principle of metric relations is implicit in the notion of this manifold, it must come from somewhere else in the case of a continuous manifold. Either then the actual things forming the groundwork of a space must constitute a discrete manifold, or else the basis of metric relations must be sought for outside that actuality, in colligating forces that operate upon it.

“A decision upon these questions can be found only by starting from the structure of phenomena that has been approved in experience hitherto, for which Newton laid the foundation, and by modifying this structure gradually under the compulsion of facts which it cannot explain. Such investigations as start out, like this present one, from general notions, can promote only the purpose that this task shall not be hindered by too restricted conceptions, and that progress in perceiving the connection of things shall not be obstructed by the prejudices of tradition.

“This path leads out into the domain of another science, into the realm of physics, into which the nature of this present occasion forbids us to penetrate.”
but rather their obtaining or not obtaining must be stipulated. On the other hand, if space possesses a discrete topological structure, metrical relations are not conventional but real.

The importance of this theory of metrical relations consists in the fact that it forms the basis for Grünbaum's conventionalism. That is, relative to the contingent truth that space is continuous, this theory, rather than the thesis that any description may be maintained, given suitable alterations in the relevant laws and auxiliary statements, is held to establish the following: (i) "[T]he alternative between different metrics and hence between their associated metric geometries is one of mere descriptive convenience" ([2], page 242). (ii) Given two descriptions such that II asserts a "unitary change in those functional dependencies or laws of nature which involve variables ranging over lengths [and distances, and] I asserts that all metersticks and extended objects [as well as the distances between them] have expanded by doubling; [I and II] are complete[ly] equivalent[t] or co-legitimate[s] . . . with regard to truth value and inductive (though not descriptive) simplicity" ([2], pages 158, 161, 173).

The purpose of what follows is to show that whether or not metrical relations are conventional is independent of the type of consideration that

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is suggested by Grünbaum's theory of metrical relations. Of the two theses which this theory is held to establish, only (i) will be explicitly discussed. The discussion will be further restricted to the space metric of nonrelativistic physics. Thus, there will be no explicit mention of the metrics of time and space-time. These restrictions, however, are purely expository, and hence any conclusion reached concerning space is readily extendible to nonrelativistic time, and, in the case of relativity, to space-time. Similarly, though (ii) will not be explicitly discussed, it follows that if Grünbaum has not established the conventionality of metrical relations, (ii) is as groundless as (i).

I

Our discussion will require the notion of a distance function or metric. This is simply a mapping which associates with any two points x, y a nonnegative real number, and which satisfies the conditions (a) $d(x,y) = d(y,x)$; (b) $d(x,y) = 0$ if and only if $x = y$; (c) $d(x,y) + d(y,z) \geq d(x, z)$. Given the distance function the class of straight intervals is determined. Given any interval [x,y] the class of intervals with which it is congruent—i.e., the congruence class to which it belongs—is given by the distance function. That is, relative to a given distance function, two closed intervals are congruent or belong to the same congruence class if their end points are associated with the same real number. Thus, the distance function or metric associates with each pair of points an interval, and with each interval, a congruence class.

Riemann's assertion that metrical relations are not implicit in the concept of a continuous space may be stated more exactly as follows: If space is continuous, there is an infinite number of possible metrics and, moreover, there is no topologically invariant property (or relation) on which the selection of one of these metrics may be based. Of course, this assertion has nothing to do with the choice of scale, or rather, given the selection of an appropriate scale, the assertion still holds. It therefore implies that there are infinitely many possible metrics which differ significantly in the sense that any two of these metrics will differ with respect to the congruence class with which at least one interval is associated.

The basis for Grünbaum's claim that in a nondenumerable, dense space, the choice among alternative metrics is conventional consists in this:

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*With regard to (i), Grünbaum claims to "have put forward positive structural reasons à la Riemann" ([2], p. 242); while (ii) is held to be "a consequence of a significant property of physical space, i.e., of its being a mathematical continuum of like elements" ([2], p. 180).
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Given any two (nondegenerate) disjoint intervals of a nondenumerable and dense space, whether or not they are congruent depends on the stipulation that some (possibly the same) pair of disjoint intervals are or are not congruent, since in this type of space, the congruence of disjoint intervals is not a real relation. The qualification that the intervals be disjoint is necessary, since, given two closed intervals such that one is a proper part of the other, we may say that they are incongruent, whatever the form of d. Having noted this, no misunderstanding should result if, henceforth, I omit explicit mention of this qualification.

Finally, the thesis that the metric is conventional is expressible as a consequence of the following two ideas: (1) that there is a plurality of significantly different possible metrics, and (2) that congruence is not a real relation. That is, given the conventionality of congruence, (1) implies that the choice among alternative metrics is conventional. Notice that by itself (1) is not sufficient for the conventionality of the metric. (By itself, (1) is at most necessary, since without it the thesis is vacuously satisfied.) This may be made clear if we leave, temporarily, our exposition of Grünbaum’s theory and consider, instead, the following criticism which might be raised against it.7

Let \( f \) represent some admissible property of a physical system \( S \) consisting of \( N \) particles, where \( f \) is some suitable real-valued function of the \( 6N \) position and momenta coordinates associated with the state \( s \) of \( S \). Then, even if the space and time metrics are held constant, the class \( G \) of all similarly suitable real-valued functions \( g \) which differ from \( f \) and from each other in the value associated with at least one \( s \) is infinite. Moreover, there is a subclass \( G^* \) of \( G \) whose members differ significantly from \( f \), and which is also infinite. Thus, Grünbaum’s claim that the metric is conventional is trivial if it is restricted to metrics which “differ” only insofar as they are based on different scales. And it is false if the metrics are meant to differ significantly, since, by parity of reasoning, it would follow that the congruence of phase space intervals is also conventional. However, given the existence of \( G^* \), all that we are required to admit is that \( f \), unlike any \( g \) in \( G^* \), reflects changes in the state of \( S \) in accordance with the inductively simplest laws.

For this criticism to be valid, it is necessary that the theory be inter-pret ed as asserting that (1) alone (i.e., without (2)) implies the conventionality of the metric. Now when we inquire after the basis for (2), certain considerations may be advanced which suggest a similar interpretation. For example, the conventionality of congruence is in turn based on the fact that space is a continuous manifold. Prima facie this seems to mean that congruence is conventional in any nondenumerable and dense structure. But then the congruence of phase space intervals must also be conventional according to Grünbaum’s theory, and if so, the theory is clearly false.

From these considerations it follows that the conventionality of the metric cannot be based solely on the possibility of a plurality of significantly different metrics, and that the conventionality of congruence cannot be based solely on nondenumerability and denseness. According to Grünbaum, what is required in addition is that the elements of the continuum be intrinsically alike. Now the states of affairs represented by different phase space points differ intrinsically. For example, the states of affairs constituting a change in pressure differ with respect to the value of this magnitude. With regard to the spatial case, on the other hand, there is no nonconventional, monadic property of spatial points on which spatial congruence may be based.8

Moreover, since it is also the case that in a nondenumerable and dense space congruence cannot be based on cardinality or any other topological

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* See, for example, [2], p. 15, footnote 7 ([1], p. 415, footnote 7): “... while holding for the mathematical continua of physical space and time, whose elements (points and instants) are respectively alike both qualitatively and in magnitude, the thesis of the conventionality of the metric cannot be upheld for all kinds of mathematical continua...” Also [2], p. 33 ([1], p. 431): “... there are continuous manifolds, such as that of colors (in the physicist’s sense of spectral frequencies) in which the individual elements differ qualitatively from one another and have inherent magnitude, thus allowing for metrical comparison of the elements themselves. By contrast, in the continuous manifolds of space and of time, neither points nor instants have any inherent magnitude allowing an individual metrical comparison between them, since all points are alike, and similarly for instants. Hence in these manifolds metrical comparisons can be effected only among the intervals between the elements, not among the homogeneous elements themselves. And the continuity of these manifolds then assures the nonintrinsicality of the metric for their intervals.” And again: “In the case of space and time, intervals as such are constituted merely by the points and instants without as yet involving any metric; but the metrical attribute of pressure is indispensable ab initio to confer identity on the elements of the continuum of pressures, and—unlike points and instants—the elements of the pressure continuum each have distinctive magnitudes of their own. Hence the difference between the individual magnitudes of the elements of the pressure continuum furnishes the intervals of the latter continuum with an intrinsic metric, in contradistinction to the intervals of space and time!” ([2], p. 261). Cf. also the passage quoted in footnote 9, below.

* Cf. footnote 3, above.
7 The criticism which follows is essentially the one which appears in [3], pp. 222ff.
property of spatial intervals, Grünbaum concludes that, in this type of space, congruence is conventional.9

By a discrete space Grünbaum understands a finite set of contiguous elements, arranged, in the two-dimensional case, like the squares of a checkerboard, and such that there is no basis for supposing them capable of further subdivision. In this type of space congruence is a real relation, since it may be based on the equipollence of intervals.10

II

It is necessary that we be quite clear concerning the meaning of Grünbaum’s thesis that space is intrinsically metrically amorphous. What is involved may best be illustrated if we contrast the strong form of this thesis with a weaker version.

In its weak form the thesis is merely a circuitous way of denying that the topology of space implies a unique metric, and hence that metrical relations may be based on topological properties (or relations). Here one means by “space” a manifold of homogeneous elements, having a nonde numerable and dense topological structure. Thus, according to the weak form of the thesis, to assert the intrinsic metrical amorphousness of space is to assert that a unique metric is not capable of being based on the topology of the space manifold.

According to the strong form of the thesis, to assert the intrinsic metric amorphousness of space is to assert that the metrical structure of any metrical relation, such as the congruence of space intervals, is conventional. That is, according to the strong form of the thesis, the extension of any

9“... upon confronting the extended continuous manifolds of physical space and time, we see that neither the cardinality of intervals nor any of their other topological properties provide a basis for an intrinsically defined metric. The first part of this conclusion was tellingly emphasized by Cantor’s proof of the equicardinality of all positive intervals independently of their length. Thus, there is no intrinsic attribute of the space between the end points of a line-segment AB, or any relation between these two points themselves, in virtue of which the interval AB could be said to contain the same amount of space as the space between the termini of another interval CD not coinciding with AB. Corresponding remarks apply to the time continuum. Accordingly, the continuity we postulate for physical space and time furnishes a sufficient condition for their intrinsic metrical amorphousness” ([2], pp. 12–13; [1], p. 413).

10Cf., for example, [2], pp. 153–154: “It is clear that the structure of this granular space is such as to endow it with a transport-independent metric: the congruences and metrical attributes of ‘intervals’ are intrinsic, being based on the cardinal number of space atoms, although it is, of course, trivially possible to introduce various other units each of which is some fixed integral multiple of one space atom. ... And the metric intrinsic to the space permits the factual determination of the rigidity under transport of any object which is thereby to qualify as an ‘operational’ congruence standard.”

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metrical relation must ultimately rest on an arbitrary stipulation such as the self-congruence of the metric standard.

Now, since the weak thesis excludes the possibility that a continuous space possesses a unique metrical structure, it might seem that it also excludes the possibility that there exists a unique and nonarbitrary partition of the class of all space intervals into distinct congruence classes. That is, it might be assumed that if only the weak thesis were true, then, though a particular partition of the class of all intervals may be stipulated, there is no unique partition to be discovered.11

This, however, would be a mistake. For the weak thesis excludes the possibility that spatial congruence has a unique metrical structure, if it is based on some topological property such as the cardinality of intervals. If congruence is not based on one or another topological property, then, so far as the weak thesis is concerned, it is quite possible that there exists a unique and nonarbitrary partition of the class of all spatial intervals into distinct congruence classes.

It is to be observed that this error is equivalent to assuming that the necessity for stipulating the extensions of metrical relations (i.e., the strong thesis) is a direct consequence of the fact that the topology of space does not imply a unique metric (i.e., the weak thesis). But in fact, this inference depends on the additional assumption that any (metrical) spatial relation which cannot be based on the topology is conventional.

In light of the preceding, it should be clear that only the strong form of the thesis will support Grünbaum’s claim that the choice among significantly different metrics is merely a matter of descriptive convenience. Hence, the considerations (a) that there is no nonconventional, monadic property of spatial points on which congruence may be based, and (b) that congruence cannot be based on cardinality or any other topological property in a nonenumerable and dense space—if they are relevant at all —must support the strong thesis. Thus, since (a) and (b) are put forward in support of the strong thesis, they must be interpreted as implying that the congruence of spatial intervals is nonconventional only if (a’) as with phase space intervals, it is capable of being based on some nonconventional, monadic property of the points contained in the intervals, or (b’) it is
which they are associated. No basis in either science or common sense is possessed by (c') and, if accepted, it would imply the conventionality of contingent relations altogether. And (d') is a totally arbitrary requirement for the distinctness of two relations, since there is no reason why a relation should manifest itself in terms of some property of a class of individuals. Hence we may safely conclude that neither (a) nor (d) affords the slightest support for the claim that distance, and hence the metric, is conventional, or equivalently, the nonconventionality of distance does not require that it satisfy either (c') or (d').

Since distance need not satisfy (d'), it is not at all clear that discreteness implies the nonconventionality of distance. Of course, discreteness does imply that it is possible to distinguish space intervals on the basis of cardinality. But since distance need not bear any interesting relation to the cardinality of the intervals with which it is associated, this fact about discrete spaces is not, by itself, sufficient to establish that distance is not conventional. Notice that it is not being suggested that cardinality can bear no relation to distance, only that it need not. And if there is no a priori reason for assuming that a given distance relation will only be associated with intervals of the same cardinality, it cannot be maintained that discreteness establishes the nonconventionality of distance; nor, therefore, can it be maintained that discreteness establishes the nonconventionality of congruence. This completes our discussion of Grünbaum's theory of metrical relations.

The argument of the preceding section may be summarized as follows: A distinction was first drawn between a weak and a strong form of the thesis that space is intrinsically metrically amorphous. Next, it was shown that the weak thesis, though acceptable, does not imply the strong thesis, which, on the assumption that space is continuous, Grünbaum's theory of metrical relations is held to establish. It was then shown that Grünbaum's theory must be understood as asserting that the congruence of spatial intervals is nonconventional only if (a') it is capable of being based on some nonconventional, monadic property of the points in the intervals, or (b') it is capable of being based on the cardinality of intervals. Concerning (a') and (b') it was shown, first, given that spatial intervals do not satisfy (a'), (b') is a plausible condition for classes, which, as far as is known, are mere classes; second, if neither (a') nor (b') is satisfied, this establishes the weak thesis. With respect to the first result, it was shown that (a') and (b') omit the connection of spatial congruence with the relation of

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12 Cf. [2], pp. 12–13, quoted above in footnote 9, which might plausibly be interpreted as asserting this.