The Uses of Probability and the Choice of a Reference Class

Many people suppose that there isn't any real problem in arguing from known relative frequencies or measures to epistemological probabilities. I shall argue the contrary — that there is a very serious and real problem in attempting to base probabilities on our knowledge of frequencies — and, what is perhaps more surprising, I shall also argue that this is the only serious problem in the epistemological applications of probability. That is, I shall argue that if we can become clear about the way in which epistemological probabilities can be based on measures or relative frequencies, then we shall see, in broad outlines, how to deal with rational belief, scientific inference of all sorts, induction, ethical problems concerning action and decision, and so on.

Since my main concern is to establish the seriousness of solving certain problems in basing probabilities on frequencies, and since I have no definitive solutions to those problems, I propose to work backward. First I shall characterize the probability relation (actually, it will be defined in terms of the primitive relation 'is a random member of') in a general axiomatic way adequate to the purpose of showing how such a probability concept can provide a framework for discussing various philosophical problems. In the next section of the paper, I shall discuss what several writers of very different persuasions have had to offer on this topic of the relation between measure statements and the probabilities we apply to particular cases, and then I shall attempt to characterize some of the difficulties that seem to me to have been insufficiently considered and offer my own very tentative solution to these difficulties. Finally I shall discuss a number of uses of this probability concept in three aspects of epistemology (observational knowledge, observational knowledge,
USES OF PROBABILITY

knowledge of empirical generalizations, and theoretical knowledge), in connection with the problem of choosing between courses of action whose outcomes are characterized by probabilities and utilities, and in connection with ontological and metaphysical questions.

I

The basic situation we shall consider is that in which we have a set of statements, in a given language \( L \), that we regard as practically certain and which we shall take as a body of knowledge or a rational corpus. Let us denote this set of statements by \( 'K' \). Probability will be defined as a relation between a statement, a body of knowledge \( K \), and a pair of real numbers. The language of \( K \) is assumed to be rich enough for set theory (at least including the ordinary theory of real numbers) and we assume that it contains a finite stock of extralogical predicates. Following Quine's proposal in ML, we shall suppose that there are no proper names in the language. We shall suppose that these extralogical primitive predicates include both relatively observational predicates such as 'is hot' or 'is blue' and relatively theoretical predicates such as 'the magnetic field at \( x, y, z \), is given by the vector \( v \)'. I shall have more to say about these predicates later; for the moment what is important is that the stock of these predicates is assumed fixed. Next, we assume that the meaning relations that hold among these predicates have been embodied in a set of axioms in the language. One could no doubt find a number of plausible ways of doing this. In the first place, one could find a number of different axiomatizations that yield the same set of theorems in the language. These will be regarded here as different formalizations of the same language. In the second place, there will be a number of axiomatizations, plausible in themselves, that yield different sets of theorems. These will be regarded as formalizations of different languages. Finally, the language must be tied to experience through relatively ostensive rules of interpretation; a language in which 'green' has the meaning which we ordinarily attribute to 'red' would be a different language than English. There will be ostensive rules for relations as well as for one-place predicates; thus 'being hotter than' or 'seeming to be hotter than' is quite as 'directly observable' as 'being green' or 'appearing green'. The general form of an ostensive rule of interpretation is this: "Under circumstances \( C \) (described in the psy-

263
chological part of the metalanguage, or simply exhibited in a general way as 'like this'), $\phi$ holds (or holds with a certain probability)." There are all kinds of problems here that a serious epistemology would have to grapple with; but for our present purposes it suffices to suppose that there is some set of relatively ostensive rules of interpretation that is characteristic of a language.

Thus the language $L$ may be characterized as an ordered quadruple $<V,T,M,O>$, where $V$ is the set of well-formed expressions of the language, $T$ the set of theorems of the language that depend only on the meaning of the logical constants 'e', $M$ the set of theorems that depend on the meaning of the extralogical constants of the language, and $O$ the set of statements of the language that can be potentially established (with certainty or with probability) on the basis of "direct observation." We assume that each of $V$, $T$, $M$, and $O$ admits of a recursive enumeration. Two languages $<V,T,M,O>$ and $<V',T'M',O'>$ are essentially the same if and only if there is an isomorphism $m$ from $V$ to $V'$, such that $S \in T$ if and only if $m(S) \in T'$, $S \in M$ if and only if $m(S) \in M'$, and $S \in O$ if and only if $m(S) \in O'$. Thus a language similar to English with the exception that 'red' and 'green' were interchanged in meaning would be essentially the same as English.

Since we shall be concerned with logical relationships both in the object language and in the metalanguage, certain notational conventions will come in handy.

Capital letters, 'S', 'T', 'R', etc., will be used to denote statements of the object language. If $S$ is a statement of the object language, 'nS' will denote the statement of the object language consisting of a negation sign, followed by $S$. The expression nS will be called the negation of $S$. If $S$ and $T$ are statements of the object language, 'S cd T' will denote the statement of the object language consisting of $S$, followed by a horseshoe, followed by $T$, the whole enclosed in parentheses; $S cd T$ will be called a conditional, $S$ is its antecedent, and $T$ is its consequent. If $S$ and $T$ are statements of the object language, 'S cj T' will denote their conjunction, and 'S al T' will denote their alternation, understood as above; similarly, 'S b T' will denote the biconditional whose terms are $S$ and $T$.

We shall also be concerned with certain terms of the object language. Lowercase and capital letters from the front end of the alphabet will be used to denote terms of the object language; if $a$ and $A$ are terms of the
USES OF PROBABILITY

object language, ‘a e A’ will be the expression of the object language formed by concatenating a, epsilon, and A. In general, terms expressed by lowercase and capital letters from the beginning of the alphabet will be assumed not to contain free variables. Thus, in general, ‘a e A’ will be a closed sentence of the object language.

Two classes of mathematical terms that are of particular interest to us are those denoting natural numbers and those denoting real numbers. There are, of course, a lot of ways in the object language of denoting a given number; thus the number 9 could be denoted by ‘9’, by ‘the number I am thinking of now’, ‘the number of planets’, etc. For our purposes here, however, we wish to restrict our attention to numerical terms expressed in some canonical form, say as an explicit decimal, i.e., a decimal given by a purely mathematical rule, where this is construed as a function whose domain is natural numbers, whose range is the natural numbers between 0 and 9 inclusive, which can be effectively calculated, and which can be expressed by a formula containing no nonlogical constants. Such real number terms in canonical form will be denoted by lowercase letters p, q, r, etc., as will the numbers themselves. Natural number terms in canonical form will be denoted by lowercase letters m, n, etc. On those occasions on which we may wish to speak of the occurrence of particular number-denoting expressions, like ‘1’, ‘2’, ‘0.5000 . . . ’ etc., we shall make use of single quotation marks.

The ambiguity of the variables p, q, etc., must be kept in mind. This ambiguity is very natural and useful in the characterization of probability that follows, but it is permissible only because we have committed ourselves to admitting only number expressions in canonical form as values of these variables. Thus we can only say that the probability of something is p (the real number p) when a certain expression involving p (the numerical expression p in canonical form) occurs in a certain set of expressions.

If p and q are real number expressions in canonical form, then ‘p gr q’ denotes the expression (statement, in fact) consisting of p, followed by the inequality sign, followed by q. Thus to write Thm ‘p gr q’ is to assert that the statement consisting of p followed by the inequality sign ‘>’ followed by q is a theorem. Some statements of the form p gr q will be undecided or undecidable, of course; but such statements will turn out not to be of interest to us. We will never be interested in statements of the following form: the proportion of A’s that are B’s is p,
where $p$ is the infinite decimal consisting of 8's if Fermat's theorem is true and of 7's if it is not true.

If $A$ and $B$ are any terms whatever (including natural number expressions and real number expressions in canonical form) then $A \equiv B'$ will denote the expression in the object language consisting of $A$, followed by the identity sign ‘$=$’, followed by $B$. If $A$ and $B$ are class expressions, $A \cap B$ is the class expression consisting of $A$, followed by the intersection sign, followed by $B$; $A \cup B$ is the class expression consisting of $A$, followed by the union sign, followed by $B$; and $A - B$ is the class expression consisting of $A$, followed by the difference sign, followed by $B$. If $A$ and $B$ are class expressions, $A \subseteq B$ is the statement consisting of $A$, followed by the inclusion sign ‘$\subseteq$’, followed by $B$.

The most important kind of statement with which we shall be concerned is what I shall call a statistical statement. Statistical statements are expressed in ordinary and scientific language by a variety of locutions: “51 percent of human births result in the birth of boys”; “the evidence shows that half the candidates can't even read English adequately”; “the odds against heads on a toss of this coin are about even”; “the distribution of foot sizes among recruits is approximately normal with a mean of 9 inches and a standard deviation of 1.3 inches.” We shall take as a standard form for statistical statements the simple sort: “The proportion of potential or possible $A$'s that are $B$'s lies between the limits $p_1$ and $p_2$,” or “The measure, among the $A$'s, of the set of things that are $B$'s, lies between $p_1$ and $p_2$.” For most purposes we need consider only statements of this form, since even when we are dealing with the distribution of a random quantity, such as foot size, or of an original $n$-tuple of random quantities, when it comes to making use of that knowledge, we will be interested in the probability that a member of the reference class belongs to the set of objects having a certain property; the distribution functions are so much mathematical machinery for arriving at statistical statements of the straightforward form in question.

Such statements will be expressed in the object language with the aid of a new primitive term, ‘%’; it is a four-place relation, occurring in contexts of the form $\% (x, y, z, w)$, where $x$ and $y$ are classes, $z$ and $w$ real numbers. $\% (x, y, z, w)$ means that the measure, in the set $x$, of those objects which belong to $y$, lies between $z$ and $w$. This statement will be named in the metalanguage by the expression $M('x', 'y', 'z', 'w')$. Put
USES OF PROBABILITY

otherwise: $M(A,B,p,q)$ denotes that statement in the object language consisting of ‘%’, followed by ‘(’, followed by the term $A$, the term $B$, followed by the real number expressions $p$ and $q$ in canonical form. If $p$ and $q$ are not expressions in canonical form denoting real numbers, then we take $M(A,B,p,q)$ to denote “0 = 1.”

We are now in a position to begin to characterize the set of statements $K$ that we will call a rational corpus or body of knowledge, to the extent that this is necessary for the axiomatic characterization of probability. For reasons which have been discussed at length elsewhere,¹ we shall not suppose that $K$ is deductively closed or consistent in the sense that no contradiction is derivable from it. We will make only the bare minimum of assumptions about $K$: namely, that it contains no self-contradictory statement and that it contains all the logical consequences of every statement it contains. To be sure, this is already asking a lot in a sense; it is asking, for example, that all logical truths be included in the rational corpus $K$. But although nobody knows, or could know, all logical truths, in accepting a certain system of logic and a certain language, he is committing himself to all these logical truths. Then why not complete logical closure? Because whereas he accepts a language and its logic all at one fell swoop, with a single act or agreement, he accepts empirical statements piecemeal, one at a time. Furthermore, although the logical truths cannot be controverted by experience, the empirical statements he can accept, with every inductive right in the world, can be controverted, and some of them no doubt will be. The man who never has to withdraw his acceptance of a factual statement has to be kept in a padded cell. Thus we state as axioms:

**Axiom I**  
$S \in K \land \text{Thm } S \vdash T \iff T \in K.$

**Axiom II**  
$(\exists S) (\sim S \in K).$

In stating the weak consistency of $K$ by axioms I and II, we are, of course, presupposing a standard logic. Furthermore, axioms I and II entail that this standard logic is consistent. Since we are supposing not only a first order logic, but a whole set theory, this consistency is not demonstrable. Nevertheless we always do suppose, so long as we know no better, that the language we speak is consistent. And if we come to know better, we will change our language.

The basic fact about probability that I shall stipulate is that every

Henry Kyburg

probability is to be based on a known statistical statement. It is, of course, precisely the burden of part of this paper to show that this is plausible. Statements that are not of the simple membership form a e A—the only form that lends itself directly to this statistical analysis—require to be connected in some fashion to statements that are of that form. The obvious way of making this connection is to use the principle that equivalent statements are to have the same probability. This equivalence principle has been criticized in recent years by Smokler, Hempel, Goodman, and others; nevertheless the arguments that have been presented against it have focused on its plausibility or implausibility in a system of Carnapian logical probability and on its connections with various natural principles of qualitative confirmation. Since we do not have here a Carnapian form of logical probability, and since we are not at all concerned with qualitative concepts of confirmation, those arguments are beside the point. I shall therefore make the natural assumption that if we have reason to believe that S and T have the same truth value (i.e., if we have reason to accept the biconditional S b T) then S and T will have the same probability. Furthermore, to say that a statement S has the probability (p,q) entails that S is known to be equivalent to a statement of the form a e A, that a e B is known, and that M(B,A,p,q) is known. It entails some other things, too, in particular that the object denoted by a be a random member of the set denoted by B.

Of course it follows logically from the fact that the measure of x in y lies between ½ and ¾ that it also lies between 1/5 and 4/5; but this latter fact is not of interest to us. Indeed, it is a nuisance to have to keep track of it. Therefore when we speak of measure statements belonging to K, we shall mean the strongest ones in K concerning the given subject matter. We shall write 'M(A,B,p,q) e_s K' to mean that M(A,B,p,q) is the strongest statement in K about A and B—i.e., that statement belongs to K, and any measure statement about A and B in K is merely a deductive consequence of that statement.

**Definition I**  
M(A,B,p,q) e_s K \iff M(A,B,p,q) e K \& M(A,B,p',q') e K \supset \text{Thm} M(A,B,p,q) \phi M(A,B,p',q').

Randomness is customarily defined in terms of probabilities; here we shall adopt the opposite course and take the probability relation to be definable in terms of randomness. The basic form of a statement
USES OF PROBABILITY

of randomness is the following: the object a is a random member of the set A, with respect to membership in the set B, relative to the rational corpus K. We shall express the metalinguistic form of this statement by ‘Ran$_K$(a,A,B)’.

Two obvious axioms are as follows:

AXIOM III Ran$_K$(a,A,B) ⊃ a ∈ A ∈ K.

AXIOM IV Ran$_K$(a,A,B) ⊃ (∃p) (∃q) (M(A,B,p,q) ∈ K).

Since we have not taken K to be deductively closed, it is perfectly possible that we should have the statements S b T and T b R in K, without having the statement S b R in K. Nevertheless, we want to be able to take account of the relationship between S and R. We shall say that they are connected by a biconditional chain in K. In general, if we have a chain of statements, S b T$_1$, T$_1$ b T$_2$, . . . , T$_n$ b R, each member of which is a member of K, we shall say that S and R are connected by a biconditional chain in K.

DEFINITION II S b c$_K$ T$'$ = at (X) ((R) (S) (T) ((R b S ∈ K ⊃ R b S ∈ X) & ((R b S ∈ X & S b T ∈ X) ⊃ R b T ∈ X)) ⊃ S' b T' ∈ X)

In order to establish the uniqueness of probabilities (clearly a desideratum), we do not need to stipulate that there be only one set of which a given object is a random member with respect to belonging to another set; it suffices that our statistical knowledge about any of the sets of which a is a random member with respect to B (relative to the rational corpus K) be the same. Thus we have the following axiom characterizing the randomness relation.

AXIOM V (a' b' c$_K$ a' B') & Ran$_K$(a,A,B) & Ran$_K$(a',A',B') & M(A,B,p,q) ∈ K ⊃ M(A',B',p,q) ∈ K.

We shall also stipulate that if a is a random member of A with respect to B, it is also a random member of A with respect to the complement of B:

AXIOM VI Ran$_K$(a,A,B) ⊃ (Ran$_K$(a,A,B min B) & Ran$_K$(a,A,A)).

We now define probability in terms of randomness: to say that the probability of S is (p,q) is to say that we know that S is equivalent to
Henry Kyburg

some membership statement $a \in B$, that $a$ is a random member of $A$ with respect to membership in $B$, and that we know that the measure of $B$ and $A$ is $(p,q)$.

**Definition III** \[ \text{Prob}(S,p,q) = \text{at} \left( \exists a \right) \left( \exists B \right) \left( \exists A \right) \left( S \text{bc}_K a \in B \& \text{Ran}_K \left( a,A,B \right) \& M(A,B,p,q) \in \epsilon, K \right). \]

We can already prove a number of theorems about probability. Two statements connected by a biconditional chain in $K$ turn out to have the same probability. We have already made use of a part of this principle in the definition of probability, but we wish it to hold generally and not merely for pairs of statements of which one is of the form $a \in B$, where for some $A$, $\text{Ran}_K(a,A,B)$.

**Theorem I** \( (S \text{bc}_K T \& \text{Prob}_K(S,p,q)) \supset \text{Prob}_K(T,p,q). \)

**Proof:** Immediate from definitions II and III and the fact that if $S \text{bc}_K T$ and $T \text{bc}_K R$, then $S \text{bc}_K R$.

The next theorem states an even stronger version of the same fact, from which theorem III, which asserts the uniqueness of probabilities, follows directly.

**Theorem II** \( (S \text{bc}_K T \& \text{Prob}_K(S,p,q) \& \text{Prob}_K(T,p',q')) \supset \text{Thm p id p' cj q id q'}. \)

**Proof:** From theorem I and the hypothesis of the theorem, we have $\text{Prob}_K(S,p,q)$ and $\text{Prob}_K(S,p',q')$.

From definition III, there must exist terms $a, a', A, A', B,$ and $B'$ such that $\text{Ran}_K(a,A,B)$ and $\text{Ran}_K(a',A',B')$

and $M(A,B,p,q) \in \epsilon, K$ and $M(A',B',p',q') \in \epsilon, K$.

From axiom V it therefore follows that $M(A,B,p',q') \in \epsilon, K$.

By definition I, we then have $\text{Thm } M(A,B,p,q) \epsilon M(A,B,p',q')$ and $\text{Thm } M(A,B,p',q') \epsilon M(A,B,p,q)$.
USES OF PROBABILITY

Since these measure statements belong to \( K \), they are not simply \( "1 = 0" \); thus in order for the conditionals to be theorems, we must have \( p \ id \ p' \) and \( q \ id \ q' \) as theorems, and thus

\[
\text{Thm } p \ id \ p' \ CJ \ q \ id \ q'.
\]

**Theorem III**

\[
(\text{Prob}_K(p,q) \& \text{Prob}_K(p',q')) \supset \text{Thm } p \ id \ p' \ CJ \ q \ id \ q'.
\]

**Proof:** Clear from the fact that \( S \ bc_K S \).

**Theorem IV**

\[
\text{Prob}_K(p,q) \supset \text{Prob}_K(nS, 1 - q, 1 - p), \text{ where } '1 - q' \text{ denotes the real number expression consisting of '1' followed by a minus sign, followed by q.}
\]

**Proof:** If \( p \) and \( q \) are real number expressions in canonical form, so are \( 1 - q \) and \( 1 - p \). \( M(A,B,p,q) cd M(A,A \min B, 1 - q, 1 - p) \) is a theorem of measure theory. The theorem now follows from axiom VI and definition III.

**Theorem V**

\[
(S \ bc_K a \ e B \ & \ \text{Ran}_K(a,A,B) \ & (\exists X) (S \ CJ T \ bc_K a \ e B \ \text{int} X \ & \ \text{Ran}_K(a,A,B \ \text{int} X)) \ & \ \text{Prob}_K(S,p,q) \ & \ \text{Prob}_K(S \ CJ T,r,s)) \supset \text{Thm } p \ g \ r \ al \ p \ id \ r.
\]

**Proof:** By definition III, and axiom V,

\[
M(A,B \ \text{int} X,r,s) \in K.
\]

By a theorem of measure theory,

\[
\text{Thm } M(A,B \ \text{int} X,r,s) \ cd M(A,B,r,'1').
\]

By axiom I, therefore,

\[
M(A,B,r,'1') \in K.
\]

But by the hypothesis of the theorem and definition III,

\[
M(A,B,p,q) \in K.
\]

And so by definition I,

\[
\text{Thm } M(A,B,p,q) \ cd M(A,B,r,'1').
\]

And thus

\[
\text{Thm } p \ g \ r \ al \ p \ id \ r.
\]

**Theorem VI**

\[
(\text{Thm } S \ cd T \ & \ \text{Prob}_K(S,p,q)) \supset \text{Prob}_K(S \ CJ T,p,q).
\]

271
Proof: Thm S cd T ⊃ Thm S b T cj S; theorem I.

THEOREM VII (Thm S cd T & Prob_κ(S,p,q) & Prob_κ(T,r,s)) ⊃ Thm r id p al r gr p.

Proof: Theorems V and VI.

THEOREM VIII (S bc_κ a e B & Ran_κ(a,A,B) & (∃X) (S al T bc_κ a e B un X & Ran_κ(a,A,B un X)) & Prob_κ(S,p,q) & Prob_κ(S al T,r,s)) ⊃ Thm s id q al s gr q.

Proof: From theorem V.

Under certain special circumstances concerning randomness we can obtain, on the metalinguistic level, a reflection of the usual probability calculus generalized to take account of interval measures. Circumstances of the kind described occur in gambling problems, in genetics, in experiments in microphysics, in the analysis of measurement, etc. So they are not as special as they might at first seem.

THEOREM IX Let {S_i} be a set of statements of L such that

(S) (S e {S_i} ⊃ (∃p) (∃q) (Prob_κ (S,p,q))).

(i) (S) (S e {S_i} ⊃ (∃p) (∃q) (Prob_κ (S,p,q))).

(ii) (S) (S e {S_i} & Prob_κ(S,p,q) ⊃ Prob_κ(nS,1-q,1-p)).

(iii) (S) (T) ((S, T e {S_i} & S bc_κ a e X & T bc_κ a e Y & X int Y int A id 'φ' e K & M(A,X,p_s,q_s) cj M(A,Y,p_t,q_t) e K & Prob_κ(S,p_s,q_s) & Prob_κ(T,p_t,q_t) & S al T e {S_i} & Prob_κ(S al T,r,s)) ⊃ Thm(r gr p_s + p_t al r id p_s + p_t) cj (q_s + q_t gr s al q_s + q_t id s)).

A strengthening of the hypothesis of this theorem will lead to a meta-
linguistic reflection of the standard calculus of probability.

THEOREM X Let K be closed under deduction, and let {B_i} be a set of terms; let the sets denoted by the terms B_i be a field of subsets of the set denoted by A. Suppose that
USES OF PROBABILITY

(a) \( (X) \{ X \in \{ B_i \} \supset (\exists p_i) (M(A,B_i,p_i,p_i) \in K) \} \) and
(b) \( (S) \{ S \in \{ S_i \} \supset (\exists X) (X \in \{ B_i \} & \text{Ran}_K \)
\( a, A, X \) & \text{S bc}_K a e X \& a e A \in K \) \).

Then: \( (i) (S) \{ S \in \{ S_i \} \supset (\exists p) (\text{Prob}_K(S,p,p)) \).
\( (ii) (S) \{ S \in \{ S_i \} & \text{Prob}_K(S,p,p) \supset \text{Prob}_K(nS, 1 - p, 1 - p) \} \).
\( (iii) (S) (T) \{ S, T, \in \{ S_i \} \& \text{S bc}_K a e B_i \& \text{T bc}_K a e B_j \& B_j \text{int} A \text{id 'p'} \epsilon \text{K} \supset (\text{Prob}_K(S,p,p) \& \text{Prob}_K(T,q,q) \supset \text{Prob}_K(S \text{al} T, p + q, p + q)) \} \).

(iv) \( (S) \{ S \text{bc}_K a e A \supset \text{Prob}_K(S, '1', '1') \} \).

Two more theorems may come in handy:

**Theorem XI** \( (\exists a) (\exists A) (\exists B) (\text{Ran}_K(a,A,B) \& S \in K) \supset \text{Prob}_K(S, '1', '1') \).

**Proof:** By axiom III, \( a e A \in K \).

Since \( S \text{cd} (a e A b S \text{cj} a e A) \) is a theorem, the bi-conditional \( a e A b S \text{cj} a e A \) belongs to K.

Since \( a e A \text{cd} (S b S \text{cj} a e A) \) is a theorem, the bi-conditional \( S b S \text{cj} a e A \) belongs to K. Thus
\( S \text{bc}_K a e A \).

By axiom VI,

\( \text{Ran}_K(a,A,A) \).

Since \( M(A,A, '1', '1') \) is a theorem, so is \( S \text{cd} M(A,A, '1', '1') \), and thus
\( M(A,A, '1', '1') \in K. \)

By definition III,

\( \text{Prob}_K(S, '1', '1') \).

**Theorem XII** \( (\exists a) (\exists A) (\exists B) (\text{Ran}_K(a,A,B)) \& S \in K \supset \text{Prob}_K(nS, '0', '0') \).

**Proof:** Theorem XI and theorem IV.

II

How are we to construe randomness? Most writers on probability and statistics have no difficulty with the concept of randomness, because they think they can define it in terms of probability. In a certain sense this is so: one may say that \( a \) is a random member of the class \( C \) if \( a \)
is selected from $C$ by a method which (let us say) will result in the choice of each individual in $C$ equally often in the long run. This is a statistical hypothesis about the method $M$, of course. It might better be expressed by the assertion that $M$ is such that it produces each member equally often in the long run. We may or may not, in particular cases, have reason to accept this hypothesis. In any event, it applies only in exceptional cases to applications of our statistical knowledge. There is no way in which we can say that Mr. Jones—the 35-year-old who took out insurance—is selected for survival during the coming year by a method which tends to select each individual with equal frequency. If there is a method, which many of us doubt, it is a method which is presumably not indifferent to the virtues and vices of the individual involved. Furthermore, even if we did know that $A$ was selected from $C$ by an appropriate random method, whether this knowledge, combined with the knowledge of $p$ of the $C$'s are $B$'s, would yield a probability of $p$ that $A$ is a $B$ would depend on what else we knew about $A$. We could know these things and also know that in point of fact $A$ was a $B$; then we would surely not want to say that the probability of its being a $B$ was $p$. Similarly, even under the most ideal circumstances of the application of this notion of randomness, we could encounter information which would be relevant to the probability in question. Thus suppose that the probability of survival of a member of the general class of 35-year-old American males is .998. Let us assign a number to each American male of this age and choose a number by means of a table of random numbers. It is indeed true that this method would tend, in the long run, to choose each individual equally often. But if the method results in the choice of Mr. S. Smith, whom we know to be a wild automobile racer, we would quite properly decline to regard his probability of survival as .998. Finally, the next toss of a certain coin, although it is the toss that it is, and no other, and is chosen by a method which can result only in that particular toss, is surely, so far as we are concerned now, a random member of the set of tosses of that coin; and if we know that on the whole the coin tends to land heads $\frac{2}{3}$'s of the time, we will properly attribute a probability of $\frac{2}{3}$ to its landing heads on that next toss.

Probability theorists and statisticians have tended to take one of two tacks in relation to this problem; either they deny the significance of probability assertions about individuals, claiming that they are mean-
USES OF PROBABILITY

...ingless; or they open the door the whole way, and say that an individual may be regarded as a member of any number of classes, that each of these classes may properly give rise to a probability statement, and that, so far as the theory is concerned, each of these probabilities is as good as another. Thus if we happen to know that John Smith is a 35-year-old hairless male Chicagoan, is a lawyer, and is of Swedish extraction, the first group would say that 'the probability that John Smith is bald is p' is an utterly meaningless assertion; and the second group would say that if, for example, we know that 30 percent of the male lawyers in Chicago are bald, and 10 percent of the males, and 5 percent of Chicagoans, and 3 percent of the males of Swedish extraction, then all the probability statements that follow are correct.

The probability that John Smith is bald is 0.30.
The probability that John Smith is bald is 0.10.
The probability that John Smith is bald is 0.05.
The probability that John Smith is bald is 0.03.
The probability that John Smith is bald is 1.00.

In either case, however, we are left with a problem which is in all essentials just the problem I want to focus on. If we say that no probability statement can be made about the individual John Smith, then I still want to know what rate to insure him at. No talk about offering insurance to an infinite number of people will help me, because even if I am the biggest insurance company in the world, I will only be insuring a finite number of people. And no talk about a large finite set of people will do, for the finite group of n people to whom I offer insurance is just as much an individual (belonging to the set of n-membered subsets of the set of people in that category) as John Smith himself. So what number will I use to determine his rate? Surely a number reflecting the death rate in some class—or, rather, reflecting what I believe on good evidence to be the death rate in some class. What class? Tell me what class to pick and why, and my problem concerning randomness will have been solved. The same is true for those who would tell me that there are any number of probabilities for John's survival for the coming year. Fine, let there be a lot of them. But still, tell me which one to use in assigning him an insurance rate.

Some philosophers have argued that this is merely a pragmatic problem, rather than a theoretical one. The scientist is through, they say, when he has offered me a number of statistical hypotheses that he re-
Henry Kyburg

gards as acceptable. Choosing which hypothesis to use as a guide to action is not his problem, but mine, not a theoretical problem, but a practical one.

But calling a problem practical or pragmatic is no way to solve it. If I am a baffled insurance company before the christening, I shall be a baffled insurance company after it.

Some of those who regard the problem of choosing a reference class as a practical problem rather than an interesting theoretical one, nevertheless have advice to offer. Some statisticians appear to offer the advice (perhaps in self-interest), 'Choose the reference class that is the most convenient mathematically,' but most suggest something along the lines of Reichenbach's maxim: 'Choose the narrowest reference class about which you have adequate statistical information.' "If we are asked to find the probability holding for an individual future event, we must first incorporate the case in a suitable reference class. An individual thing or event may be incorporated in many reference classes . . . We then proceed by considering the narrowest reference class for which reliable statistics can be compiled." Reichenbach recognizes that we may have reliable statistics for a reference class A and for a reference class C, but none for the intersection of A and C. If we are concerned about a particular individual who belongs both to A and to C, "the calculus of probability cannot help in such a case because the probabilities \( P(A,B) \) and \( P(C,B) \) do not determine the probability \( P(A \cap C, B) \). The logician can only indicate a method by which our knowledge may be improved. This is achieved by the rule: look for a larger number of cases in the narrowest common class at your disposal." This rule, aside from being obscure in application, is beside the point. If we could always follow the rule "investigate further," we could apply it to the very individual who concerns us (wait a year and see if John Smith survives), and we would have no use for probability at all. Probability is of interest to us because we must sometimes act on incomplete information. To be sure, the correct answer to a problem may sometimes be—get more information. But from the point of view of logic this answer can be given to any empirical question. What we expect from logic is rather an indication of what epistemological stance we should adopt with respect to a given proposition, under given cir-

USES OF PROBABILITY

circumstances. Perhaps in the situation outlined by Reichenbach, our answer would be, "Suspend judgment." That would be a proper answer. But since, as we shall see, every case can be put in the form outlined by Reichenbach, that answer, though relevant, seems wrong.

Surprisingly, Carnap does not give us an answer to this question either. The closest he comes to describing the sort of situation I have in mind in which we have a body of statistical information and we want to apply it to an individual is when the proposition e consists of a certain structure-description, and h is the hypothesis that an individual a has the property. P. The degree of confirmation of h on e will be the relative frequency of P in the structure-description e. This does not apply, however, when the individual a is mentioned in the evidence—i.e., when the evidence consists of more than the bare structure description. This is simply not an epistemologically possible state of affairs. Hilpinen, similarly, supposes that if we have encountered a at all, we know everything about it. This is the principle of completely known instances. It follows from this principle that if we know that a belongs to any proper subset of our universe at all, we know already whether or not it belongs to (say) B. In real life, however, this situation never arises. Even Reichenbach's principle is more helpful than this.

To make precise the difficulty we face, let us return to the characterization of a body of knowledge in which we have certain statistical information, i.e., a body of knowledge in which certain statistical statements appear. The exact interpretation of these statements can be left open—perhaps they are best construed as statements of limiting frequencies, as propensities, or as certain characteristics of chance setups, or abstractly as measures of one class in another. In any event, I take it to be the case that we do know some such statements. I shall further suppose that they have the weak form, the measure of R's among P's lies between $p_1$ and $p_2$: $\% (P, R, p_1, p_2)$.

Three ways in which we can come to know such statements—neither exclusive nor exhaustive—are the following:

(a) Some such statements are analytic: the proportion of 1000-member subsets of a given set that reflect, with a difference of at most .01, the proportion of members of that set that belong to another given set lies between .90 and 1.0. Such statements are set-theoretically true

of proportions and limiting frequencies; or true by virtue of the axioms characterizing 'measure' or 'propensity'; in either case derivable from an empty set of empirical premises.

(b) Some such statements are accepted on the basis of direct empirical observation: the proportion of male births among births in general is between 0.50 and 0.52. We know this on the basis of established statistics, i.e., by counting.

(c) Since we know that having red or white flowers is a simply inherited Mendelian characteristic of sweet peas, with the gene for red flowers dominant, we know that the proportion of offspring of hybrid parents that will have red flowers is 3/4. This statistical statement, like many others, is based on what I think is properly called our theoretical knowledge of genetics.

There are a number of expressions that one would never take to refer to reference classes. One would never say that the next toss of this coin is a random member of the set consisting of the union of the unit class of that toss with the set of tosses resulting in heads. The fact that nearly all or perhaps all of the tosses in that class result in heads will never be of relevance to our assessment of probabilities. Neither is the fact that practically all the members of the union of the set of tosses yielding tails with the set of prime numbered tosses are tosses that yield tails ever relevant to our expectations. Similarly not all properties will be of concern to us: the grueness of emeralds is not something we want to bother taking account of. This may be regarded as a commitment embodied in the choice of a language. We may, for example, stipulate that probabilities must be based on our knowledge of rational classes, where the rational classes may be enumerated by some such device as this: Let $L$ be monadic with primitive predicates $P_1$, $P_2$, \ldots, $P_n$. Then the set of atomic rational classes is composed of sets of the form $\{x: P_1 x\}$ or of the form $\{x: \neg P_n x\}$. The general set of rational classes is the set of all intersections of any number of these atomic rational classes. An object will be a random member of a class, now, only if the class is a rational class. A similar arbitrary and high-handed technique may be employed to elude the Goodmanesque problems. To be sure, all the problems reappear in the question of choosing a language—e.g., choosing a language in which $P_1$ doesn't denote the union of the unit class of the next toss with the set of tosses yielding tails. But this is a somewhat different problem. It is solved partly by
USES OF PROBABILITY

the constraint that the language must be such that it can be learned on the basis of ostensive hints and partly by the constraints which in
general govern the acceptance of theoretical structures. In any event,
so far as our syntactical reconstruction of the logic of probabilistic state-
mements is concerned, the question of the source of the particular language
L we are talking about is academic.

Since what we are concerned about here is the choice of a reference
class, we may simply start with a large finite class of classes, closed
under intersection. To fix our ideas, let us focus on a traditional bag
of counters. Let the counters be round (R) and triangular (Re), Blue
(B) and Green (Be'), zinc (Z) and tin (Ze'). Let our statistical knowl-
edge concerning counters drawn from the bag (C) be given by the fol-
lowing table, in which all the rational reference classes appropriate to
problems concerning draws from the urn are mentioned.

<table>
<thead>
<tr>
<th>Reference class</th>
<th>Subclass</th>
<th>Lower and upper bounds of chance</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>R</td>
<td>0.48 0.52</td>
</tr>
<tr>
<td>C</td>
<td>B</td>
<td>0.60 0.70</td>
</tr>
<tr>
<td>C</td>
<td>Z</td>
<td>0.45 0.55</td>
</tr>
<tr>
<td>C ∩ R</td>
<td>B</td>
<td>0.55 0.68</td>
</tr>
<tr>
<td>C ∩ R</td>
<td>Z</td>
<td>0.50 0.65</td>
</tr>
<tr>
<td>C ∩ R</td>
<td>R</td>
<td>1.00 1.00</td>
</tr>
<tr>
<td>C ∩ B</td>
<td>R</td>
<td>0.45 0.65</td>
</tr>
<tr>
<td>C ∩ B</td>
<td>Z</td>
<td>0.30 0.50</td>
</tr>
<tr>
<td>C ∩ B</td>
<td>B</td>
<td>1.00 1.00</td>
</tr>
<tr>
<td>C ∩ Z</td>
<td>B</td>
<td>0.25 0.65</td>
</tr>
<tr>
<td>C ∩ Z</td>
<td>R</td>
<td>0.35 0.70</td>
</tr>
<tr>
<td>C ∩ Z</td>
<td>Z</td>
<td>1.00 1.00</td>
</tr>
<tr>
<td>C ∩ R ∩ B</td>
<td>Z</td>
<td>0.50 0.80</td>
</tr>
<tr>
<td>C ∩ R ∩ Z</td>
<td>B</td>
<td>0.30 0.80</td>
</tr>
<tr>
<td>C ∩ B ∩ Z</td>
<td>R</td>
<td>0.25 0.90</td>
</tr>
</tbody>
</table>

Now let us consider various counters. Suppose that we know that
a₁ ∈ C, and, in terms of our basic categories, nothing more about it.
What is the probability that a₁ ∈ R? Clearly (0.48,0.52). C is the ap-
propriate reference class, since all we know about a₁ is that it is a member
of C.

Suppose we know that a₂ ∈ C and also that a₂ ∈ R. (Perhaps a₂ is
Henry Kyburg
drawn from the bag in the dark, and we can feel that it is round.)
Surely knowing that \( a_2 \) is round should have an influence on the prob-
ability we attribute to its being blue. Since we have statistical informa-
tion about \( R \cap C \), and since \( R \cap C \) is 'narrower' than \( C \), again there
is no problem: the probability is \((0.55,0.68)\). Note that Carnap and
Hilpinen and others who hold a logical view of probability must here
abandon the attempt to base this probability strictly on statistical mea-

Suppose that we know of \( a_3 \) that it belongs to both \( C \) and \( B \). What
is the probability that it belongs to \( R \)? Again we have two possible ref-
erence classes to consider: \( C \) and \( C \cap B \). According to the broader refer-
ence class, the measure is \((0.48,0.52)\); according to the narrower one
it is \((0.45,0.65)\). It seems reasonable, since there is no conflict between
these measures, to regard the probability as the narrower interval, even
though it is based on the broader reference class.

On the other hand, suppose we ask for the probability that \( a_3 \in Z \).
There the measure in the broader class is \((0.45,0.55)\), while the mea-
sure in the narrower class is \((0.30,0.50)\). Here there does seem to be
a conflict, and it seems advisable to use the narrow reference class, even
though the information it gives is not so precise; we would tend to say
that the probability of \( 'a_3 \in Z' \) is \((0.30,0.50)\).

How about an object drawn from the urn about which we can tell
only that it is blue and round? Is it zinc? The probability that this coun-
ter (say, \( a_4 \)) is zinc will have one value or another depending on the
statistical knowledge we have. We know that between 0.45 and 0.55
of the counters are zinc, that between 0.50 and 0.65 of the round coun-
ters are zinc, that between 0.30 and 0.50 of the blue counters are zinc,
and that between 0.50 and 0.80 of the round blue counters are zinc.
Here again it seems natural to use the smallest possible class \((C \cap B
\cap R)\) as the reference class, despite the fact that we have relatively
little information about it; i.e., we know the proportion of zinc counters
in it only within rather wide limits.

Now take \( a_5 \) to be a counter we know to be blue and zinc. Here, as
distinct from the previous example, it seems most natural to take the
broadest class \( C \) as the reference class. It is true that we know some-
thing about some of the smaller classes to which we know that \( a_5 \) be-
longs, but what we know is very vague and indefinite, whereas our
knowledge about the class $C$ is very precise and in no way conflicts with our knowledge about the narrower class.

Finally, let $a_6$ be known to belong to $C \cap R \cap Z$. What is the probability of the statement `$a_6 \in B$'? This case is a little more problematic than the preceding one. One might, by an argument analogous to the preceding one, claim that the probability is $(0.60, 0.70)$: the smallest rational reference class to which $a_6$ is known to belong is one about which we have only vague information which seems not to conflict with the very strong information we have about $C$ and $B$. But there is a difference in this case: $a_6$ is also known to belong to $C \cap R$, and our information about the measure of $B$ in $C \cap R$ does conflict with our information about the measure of $B$ in $C$. If one interval were included in the other, we could simply suppose that our knowledge in that case was more precise than our knowledge in the other case. But the intervals overlap: it is not merely a case of greater precision in one bit of knowledge than in the other. Thus one might want to argue (as I argue in *Probability and the Logic of Rational Belief*) that the knowledge that $a_6$ belongs to $C \cap R$ prevents $a_6$ from being a random member of $C$ with respect to belonging to $B$, i.e., prevents $C$ from being the appropriate reference class.

The various situations described here are summarized in the following table:

<table>
<thead>
<tr>
<th>We know (altogether)</th>
<th>We are interested in</th>
<th>Possible reference class</th>
<th>Measure</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$'a_1 \in C'$</td>
<td>$'a_1 \in R'$</td>
<td>$C$</td>
<td>$(0.48,0.52)$</td>
<td>$(0.48,0.52)$</td>
</tr>
<tr>
<td>$'a_2 \in C \cap R'$</td>
<td>$'a_2 \in B'$</td>
<td>$C$</td>
<td>$(0.60,0.70)$</td>
<td>$(0.55,0.68)$</td>
</tr>
<tr>
<td>$'a_3 \in C \cap B'$</td>
<td>$'a_3 \in R'$</td>
<td>$C \cap R$</td>
<td>$(0.48,0.52)$</td>
<td>$(0.45,0.55)$</td>
</tr>
<tr>
<td></td>
<td>$'a_3 \in Z'$</td>
<td>$C \cap B$</td>
<td>$(0.45,0.55)$</td>
<td>$(0.30,0.50)$</td>
</tr>
<tr>
<td>$'a_4 \in C \cap B \cap R'$</td>
<td>$'a_4 \in Z'$</td>
<td>$C \cap B$</td>
<td>$(0.45,0.55)$</td>
<td>$(0.30,0.50)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$C \cap R$</td>
<td>$(0.30,0.50)$</td>
<td>$(0.50,0.65)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$C \cap R \cap B$</td>
<td>$(0.50,0.80)$</td>
<td>$(0.50,0.80)$</td>
</tr>
</tbody>
</table>


281
To formulate the principles implicit in the arguments that have resulted in the foregoing table, we require some auxiliary machinery. We shall say that X prevents, in $K_1$, a from being a random member of Y with respect to membership in Z—in symbols, $X \text{Prev}_{K_1}(a,Y,Z)$; when X is a rational class, $a \in X \in K_1$, there are two canonical number expressions $p_1$ and $p_2$ such that $M(X,Z,p_1,p_2) \in K_1$, for every pair of canonical number expressions $q_1$ and $q_2$, if $M(Y,Z,q_1,q_2) \in K_1$ then either $p_1 \text{gr} q_1 \text{cj} p_2 \text{gr} q_2$ is a theorem, or $q_1 \text{gr} p_1 \text{cj} q_2 \text{gr} p_2$ is a theorem, and finally, provided $Y \text{incl} X$ (the statement asserting that Y is included in X) is not a member of $K_1$. Formally:

\[
X \text{Prev}_{K_1}(a,Y,Z) \equiv (X \text{ is a rational class } \& a \in X \in K_1 \& \exists p_1 \exists p_2 (M(X,Z,p_1,p_2) \in K_1 \& (q_1)(q_2) (M(Y,Z,q_1,q_2) \in K_1 \supset \text{Thm } p_1 \text{gr } q_1 \text{cj } p_2 \text{gr } q_2 \lor \text{Thm } q_1 \text{gr } p_1 \text{cj } q_2 \text{gr } p_2)) \& \sim Y \text{incl } X \in K_1).
\]

In informal terms, the two principles that have guided the construction of the table above are as follows:

**Principle I:** If a is known to belong to both X and Y, and our statistical knowledge about X and Z differs from that about Y and Z, i.e., if one interval is not included in the other, then, unless we happen to know that X is included in Y, Y prevents X from being an appropriate reference class. If neither is known to be included in the other, then each prevents the other from being an appropriate reference class.

**Principle II:** Given that we want to avoid conflict of the first sort, we want to use the most precise information we have; therefore among all the classes not ruled out by each other in accordance with principle I, we take as our reference class that class about which we have the most precise information.

282
USES OF PROBABILITY

These two principles might be embodied in a single explicit definition of randomness:

**Definition:** \( \text{Ran}_{K_1}(a,Y,Z) \equiv X \) is a rational class & \( a \in Y \in K_1 \) & \( (X) (\sim X \text{Prev}_{K_1}(a,Y,Z)) \) & \( (X') ((X) (\sim X \text{Prev}_{K_1}(a,Y',Z)) \supset (p_1)(p_2)(q_1)(q_2)((M(Y,Z,p_1,p_2) \in K_1 \& M(Y',Z,q_1,q_2) \in K_1) \supset (\text{Thm } n(p_1 \text{ gr } q_1 \text{ cj } p_2 \text{ gr } q_2) \& \text{Thm } n(q_1 \text{ gr } p_1 \text{ cj } q_2 \text{ gr } p_2)))). \)

These principles, though they seem plausible enough in the situations envisaged above, do lead to decisions that are mildly anomalous. For example, suppose our body of knowledge were changed from that described by the first table, so that when we know that \( a \in C \cap B \cap R \), our relevant statistical knowledge is represented by:

\[
\begin{align*}
\% (C,Z,0.45,0.50) \\
\% (C \cap B,Z,0.48,0.51) \\
\% (C \cap R,Z,0.49,0.52) \\
\% (C \cap B \cap R,Z,0.20,0.90).
\end{align*}
\]

Then by the principles set forth, \( a \) is a random member of \( C \cap B \cap R \) with respect to belonging to \( Z \), and its probability is \( 0.20,0.90 \).

Various alternatives might be suggested. One would be to take the intersection of the measure intervals to be the probability in the case of conflict. But the intersection may be empty, in some cases, and even when this is not so, the intersection would give a very deceptive image of our knowledge. How about the union? The union will not in general be an interval, but we might take the smallest interval that covers the union. Thus in the last example, one might base a probability statement on both \( \% (C \cap B,A,0.48,0.51) \) and \( \% (C \cap R,A,0.49,0.52) \), taking the probability of \( a \in Z \) to be \( 0.48,0.52 \).

Another suggestion—this one owing to Isaac Levi\(^5\)—is that the set of potential reference classes be ordered with respect to the amount of information they give. In the simple case under consideration, this information is simply the width of the measure interval. We could then proceed down the list, employing principle I. The intention would be to ignore conflict, when it was a class lower in the information content ordering that conflicted with one higher in the list.

\(^6\) I. Levi, oral tradition.
What statements go into a rational corpus $K$? Well, part of what a man is entitled to believe rationally depends on what he has observed with his own two eyes. Or, of course, felt with his own two hands, heard with his own two ears (in at least some senses), or smelled with his own nose. There are problems involved with the characterization of this part of a rational corpus. The line between observation statements and theoretical statements is a very difficult one to give reasons for having drawn in one place rather than in another. It will be part of the burden of this section to show, however, that so far as the contents of $K$ is concerned, it is irrelevant where it is drawn.

Let us distinguish two levels of rational corpus; let $K_1$ denote the set of statements belonging to the higher level rational corpus, and let $K_0$ denote the set of statements belonging to the lower level rational corpus. By theorem XI, $K_1 \subseteq K_0$.

A. Observation Statements.

We wish sometimes to talk about the body of knowledge of an individual (John's body of justifiable beliefs) and sometimes we wish to talk about the body of knowledge of a group of individuals (organic chemists, physicists, scientists in general).

There are three treatments of observational data that seem to have a rationale of some degree of plausibility. Each of them, combined with an account of epistemological probability of the sort that has been outlined, seems to lead in a reasonable way to bodies of knowledge that look something like the bodies of knowledge that we actually have.

The most complex, least natural, and most pervasive treatment is a phenomenological one in which observations are regarded as indubitable and in which observations are of seemings or appearings. They may not be of appearances: I observe that I am having a tree appearance; I do not observe the appearance of a tree. Let us write $'L_P'$ for the set of sentences of our original language $L$ that correspond to what we naively call 'observation statements' or, more technically, 'physical object observation statements'. These include such statements as 'the meter reads 3.5', 'there is a dog in the house', 'there are no elephants in the garden', 'the soup is not hot', and the like. Let us write $'L_s'$ for the set of sentences in $L$ which correspond to what we regard as phenomeno-
USES OF PROBABILITY

logical reports: 'I seem to see a tree,' 'I take there to be a meter reading 3.5,' and so on. The statements of this class include (but may not comprise) statements of the form 'S seems to be the case to me', where S is a statement belonging to $L_P$. This special subclass — which may perfectly well be a proper subclass — of $L_S$ is what we shall take to include (in this view) the foundations of empirical knowledge. Let us take such statements to be of the form $J(I,t,S)$, where I is an individual or a set of individuals and t is a time parameter. Contrary to the impression one gets from most who write in this vein, there seems to be no reason why this sort of view cannot be developed directly on a multipersonal basis. To say that S appears to be the case to a set of individuals I is to say that the individuals in I collectively judge that S seems to be the case (at t). Perhaps they are all looking at the cat on the roof. These statements — statements of the form $J(I,t,S)$ — which belong to $L_S$, and thus to $L$, are indubitable, and therefore may properly occur in any rational corpus, say $K_1$, however rigorous its standards of acceptance.

We must now account for the acceptance of ordinary observation statements concerning physical objects, i.e., sentences belonging to $L_P$. On the view under consideration, sentences belonging to $L_P$ must be viewed as relatively theoretical statements — as miniature theories — and can have nothing but their probability to recommend them. From statements about physical objects (say, about my body, my state of health, the illumination of the scene, the placement of other physical objects, the direction of my attention, and the like), together with certain statements about what seems to be the case to I (that I seems to be in such and such a position, seems to be in good health, etc.), together with a fragment of psychophysical theory, certain other seeming statements follow. To put the matter slightly more formally, let $T$ be the set of statements about physical objects ($T \subseteq L_P$) and let $B$ be the set of statements about what seems to I to be his circumstances ($B \subseteq L_S$). In order to form the connection between $T$ and $B$, on the one hand, and another subset $O$ of $L_S$ (the predictions of the theory), on the other, we need to supplement the theory by a baby psychophysical theory $PT$. From the point now being considered, we must take the whole theory $PT \cup T$, which is a subset of $L$, and form the conjunction of its axioms (we suppose it finitely axiomatizable). Let us abbreviate that conjunction by $PT \& T$. From $PT \& T$, conjoined with $B$,
we obtain the subset \( O \) of \( L_s \). The statements in the set \( O \) are of the form \( J(I,t,S) \); and they may thus be definitively confirmed. Their definitive confirmation provides probabilistic confirmation of the conjunction of axioms of \( PT \) and \( T \), and thus also of the axioms of \( T \), and thus also of the elements of \( T \). But \( T \) is just the subset of \( L_P \) whose elements we were trying to find justification for. The question of probabilistic confirmation of theories is complex and incompletely understood, but it is no more difficult in this case than in others. If we may include elements of \( B \) and of \( O \) in \( K_1 \), thus may provide grounds for the inclusion of the conjunction \( PT \& T \) in \( K_0 \); and, since the elements of \( T \) are deductive consequences of the axioms of \( PT \& T \), therefore for the inclusion of elements of \( T \) in \( K_0 \). Furthermore, given the general theory \( PT \) in \( K_0 \) a single element of \( L_s \) in \( K_1 \) may suffice for the (probabilistic and tentative) inclusion of the corresponding statement \( L_P \) in \( K_0 \).

The second treatment of observational knowledge is much like the first. Again, seeming statements belonging to the set of statements \( L_s \) are all that can be known directly and indubitably. Again, physical object statements are components of theories. In this version, however, the psychophysical part of the theory, \( PT \), is regarded as analytic. The argument is that to deny that most of the time when \( I \) thinks he is seeing a cat would be to deprive the term 'cat' of its conventional meaning. It is a part of the meaning of the term 'cat' that a cat is an identifiable physical object; and that consequently most of the time when a person thinks he sees a cat, he does see a cat. Under this treatment, both axioms for \( PT \) and observations (\( I \) thinks, \( I \) sees . . . ) occur in \( K_1 \) and, with statements \( O \), support the inclusion of statements of \( T \) in \( K_0 \). For example, one of the analytic \( PT \) statements might be as follows: when a person feels awake and attentive, and thinks he sees a horse, 99 percent of the time a horse is there. Putting it more precisely, we can say that the measure of the set of times during which \( I \) feels awake and thinks he sees a horse, which are also times during which he is seeing a horse, lies close to 0.99. The time \( t_1 \) is a time when \( I \) feels awake and thinks he is seeing a horse. If the time \( t_1 \) is a random member of such times, with respect to the property in question, then the probability is about 0.99 (relative to the body of knowledge \( K_1 \)) that \( I \) is seeing a horse at \( t_1 \). And if this probability is high enough for inclusion in the body of knowledge \( K_0 \), there we are. Notice that error
USES OF PROBABILITY

is no problem here, any more than for any other kind of probabilistic knowledge. At some other time, say $t_2$, the body of knowledge $K_1$ may have changed and become the body of knowledge $K_1^*$. Relative to the body of knowledge $K_1^*$, it may not be the case that $t_1$ is a random member of the times during which I thinks he is awake and is seeing a horse, with respect to being a time when I is in fact seeing a horse. And thus at $t_2$ it may not be the case that the probability is 0.99 that I is seeing a horse at $t_1$. And thus at $t_2$ the statement that I is (was) seeing a horse at $t_1$ would no longer, under these circumstances, be included in $K_0$. Furthermore, it is not always the case, even at $t_1$, that the time $t_1$ is random in the appropriate sense. If I knows that he has just been given a hallucinogenic drug, or if I finds himself in utterly unfamiliar and dreamlike surroundings, or if I finds himself simultaneously in the presence of people who have long been dead, or ... then the time in question will not be a random member of the set of times when I feels awake and thinks he sees a horse, with respect to the times when he does see a horse; for these times are members of subsets of those times in which the frequency of horses (or of veridical perceptions in general) is far slighter than it is in the more general reference class.

The third approach to observation statements proceeds more directly and naturally, but leads to essentially the same results. According to this approach, we take observation statements to be physical language observation statements — members of $L_P$ — and take them initially at face value. To be sure, they can be in error: that is, an observation statement may have to be rejected in the face of contrary evidence. To allow for this probabilistic element, we take account of what experience has taught us: that not all our observation judgments are veridical. Which is to say, the measure of the set of times such that I judges $S$ to hold at $t$ ($J(I,t,S)$, where $S \in L_P$), which are times such that $S$ does not hold, is not zero. This is something we have learned. The statistical statement expressing it is a member of a pretty high level rational corpus, say $K_1$. Let $J(I,t_1,S)$ also belong to $K_1$. If $t_1$ is a random member of the set of times in which judgments of the sort $J(I,t,S)$ are made, with respect to being veridical, then the probability, relative to $K_1$, that $S$ holds at $t_1$, is very high, and $S$ may be included in $K_0$. Note that we do not have to lump all observation judgments together, but can treat them as being of different sorts, with different frequencies of error.
Henry Kyburg

No reference to any relation between phenomenological judgments (of the form \( J(I,t,S) \)) and physical object statements, other than the bare statistical, learned, fact that in a small proportion of cases these judgments are in error, is required.

Whichever approach to observational knowledge is adopted, there is a great deal more required for an adequate explication than is indicated by the preceding sketches. But in any event, it seems clear that (a) an epistemological concept of probability is central to any such explication regardless of which approach is adopted and (b) such a concept of probability may perfectly well be one in which probabilities are based on statistical statements, combined with a condition of randomness. The first case, in which physical object statements are construed as parts of a general theory designed to account for phenomena, is the most questionable; but we shall see shortly that this statistical-epistemological account of probability can serve in general to explicate the grounds of acceptance of theories. Anyway, what concerns us here is not the general primordial argument from phenomena to physical objects (which must, on the first view, be accounted for) but the direct and straightforward and generally veridical argument from a given physical appearance to a physical state of affairs, which is essentially the same on each of the approaches considered.

B. Empirical Laws.

There is no problem about the acceptance of statistical generalizations in \( K_0 \), given the appropriate data in \( K_1 \). At the very least in \( K_1 \) we will have such analytic measure statements as 'the measure of the set of \( n \)-membered subsets of \( X \), in which the proportion of \( p \) of \( Y \)'s is such that the difference between \( p \) and the true measure of \( Y \)'s among the set of all \( X \)'s is less than \( \varepsilon_1 \), is between \( 1 - \varepsilon_2 \) and \( 1 \). Furthermore, we may perfectly well have in \( K_1 \) (on observational grounds), '\( s \) is an \( n \)-membered sample of \( X \)'s in which the proportion of \( Y \)'s is \( p^0 \).' Under appropriate circumstances it may be the case that \( s \) is a random member of the set of \( n \)-membered subsets of \( X \) with respect to membership in the representative set described above, relative to what is known in \( K_1 \). If it is, then clearly the probability of the statement '\( p \) differs from \( p^0 \) by less than \( \varepsilon_1 \)' is \((1 - \varepsilon_2, 1)\); and if \( \varepsilon_2 \) is small enough, we will be able to include this statement in \( K_0 \). But this statement is equivalent
USES OF PROBABILITY

to $M(X,Y,p^o - \epsilon_1, p^o + \epsilon_1)$. Thus we obtain a measure statement—which is a general empirical statement—in the body of knowledge $K_o$.

Universal generalizations can be approached by statistical generalizations. For example, if in our sample all the X’s turned out to be Y’s, we would be entitled (under appropriate circumstances) to include ‘$M(X,Y,1 - \epsilon_1,1)$’ in $K_o$. The content of this statement is that practically all (at least) of the X’s are Y’s. This is still a far cry from a universal generalization: it is not equivalent, for example, to ‘practically all of the non-Y’s are non-X’s’. (This has a bearing on the traditional problem of the ravens: to examine white shoes is irrelevant to the confirmation of the statistical generalization, ‘practically all ravens are black’.) To pass from ‘practically all’ statements to an ‘all’ statement is only legitimate under certain conditions and requires a new epistemological axiom.

The content of the axiom is essentially that if it is known that practically all A’s are B’s, and if our body of knowledge is such that anything known to be an A is, relative to that body of knowledge, practically certain to be a B, and furthermore such that, at the next lower level, it is not known that there is anything that is not a B, though it is an A, then, since we will in every case be able to include that any given A is a B in the lower level rational corpus, we might as well include the universal conditional that all A’s are B’s in that lower level body of knowledge. This is embodied in the following axiom: (Let us write $eqv \phi$ for the existential quantification with respect to the object language variable $v$ of $\phi$, and $uqv$ similarly for universal quantification.)

**AXIOM C**

\[ uqv(v e X \cap v e Y) \epsilon K_o \equiv [M(X,Y,1 - \epsilon_1) \epsilon K_1 \& 1 - \epsilon \text{ is greater than the acceptance level for } K_0 \& (Z) (Z e X \epsilon K_1 \supset Z e Y \epsilon K_1 \cap \text{ Ran}_K(Z,X,Y)) \& \sim eqv(v e X \cap v e Y) \epsilon K_0]. \]

It is evident that a universal generalization can only be included in a rational corpus $K_o$ if it is already the case that every occurrence of an instance of the antecedent of the generalization in $K_1$ is accompanied by the occurrence of the corresponding instance of its consequent in $K_o$. If the observational basis of $K_1$ should change, of course, we may come to reject the universal generalization—we may find a counterexample. For that matter, given an enlargement of the observational
basis of $K_1$, we may come to reject the statistical generalization. There is no offense to fallibilism in axiom G.

C. Theories.

The question of theories is somewhat more complicated than the question of statistical or universal generalizations in our given language $L$. For in general it is the case that the introduction of a theory involves a change, or at least an extension, of our language. As is well known, it is very difficult to draw sharp lines in a theory between sentences which are to be construed as giving a partial analysis of the meaning of the theoretical terms introduced and sentences that are to be construed as embodying part of the empirical content of the theory. In adopting a theory, it is correct to say that we are adopting a particular extension of our language; it is also, appropriately construed, no doubt correct to say that we are adopting a certain conceptual scheme. So far as a body of knowledge $K_0$ is concerned, however, the distinction between empirical generalizations and meaning postulates is irrelevant. What counts so far as $K_0$ is concerned is simply whether or not a given statement is acceptable; and both generalizations that are true 'by virtue of the meanings of the term involved' and factual generalizations can perfectly well come to be accepted in $K_0$. Although there are other criteria for the acceptance of a theory — for example, we prefer a theory such that there is no stronger one or simpler one equally acceptable — one basic criterion is that the theory entail no statements that conflict with statements in $K_0$. (This harmony may sometimes be achieved by rejecting hitherto accepted members of $K_0$, as well as by being cautious about accepting theories.) No doubt some of the general statements entailed by the theory will already be members of $K_0$; others, for which there is no independent evidence, will be members of $K_0$ simply by virtue of being consequences of an acceptable theory. As to the question of whether the theory as a whole may be accepted (as opposed, that is, to the question of which of several competing theories should be accepted), again we may look at the matter in an essentially statistical manner: the theory is certainly acceptable if all of its consequences are acceptable; but to say of a theory that all its consequences are acceptable is simply a universal generalization (on a metalinguistic level) of the sort we have already considered. To be sure, matters are not so simple in the case of the consequences of
USES OF PROBABILITY

theories as they are in the case of crows; it is much easier to obtain a
sample of crows that is random in the required statistical sense than a
sample of consequences of a theory that is random in the required sense.
That is why the emphasis in sampling consequences of theories is on
sampling a wide variety of different sorts of consequences. But what is
involved here is a question of logical technique which cannot be profit-
ably discussed until we have an epistemological concept of randomness
in terms of which to discuss it.

D. Philosophical Theses.

If ontological questions are to be answered in probabilistic terms rath-
er than in a priori terms, it seems reasonable to take as a criterion of
existence the acceptability of a corresponding existential statement in a
body of knowledge. Thus on the basis of the preceding arguments it
seems plausible to say that both physical objects and sense data exist,
and to characterize these assertions by the number that represents the
level of acceptability in $K_0$, since on either approach, existential state-
ments of both sorts may occur in $K_0$. As for theoretical entities, why
should it not be reasonable to say that, at a given level of acceptance, we
are committed to those entities which exist according to the best (most
powerful, most simple ...) theory acceptable at that level?

Metaphysical assertions may be treated much like ontological ones.
For example, let us look at the question of universal causation, leaving
aside the complicated question of formulating that proposition or identi-
fying particular instances of causation. Let us assume (with most writers
on these topics) that we do in fact know of many causal relationships:
that cold, under appropriate boundary conditions, causes water to freeze;
that virus X, under appropriate conditions, causes a certain cluster
of symptoms we call 'flu'; and so on. Let us further suppose, also in
accordance with most writers, that we do not know of any event that
is, in point of fact, not caused. Starting from these premises, some writ-
ers have concluded that the law of universal causation is very probable;
others have concluded that all we can say is that some events are caused
by others. If we have a frequency based epistemological concept of prob-
ability, can we adjudicate this dispute?

The answer is yes. We can reconstruct the argument as follows: in all
the cases about which we have knowledge, the law of causation operates.
Henry Kyburg

Regard these cases as a sample of all cases whatever. It is a logical truth (alluded to earlier) that practically all large samples from a class closely reflect the measure of a specified subclass in that class. Let both of these statements belong to that body of knowledge $K_1$. Assuming that 'practically all' is represented by a large enough number to represent acceptability in the rational corpus $K_0$, what more is needed? We also need to be able to say that the sample is a random member of the set of samples of that size, from this population, with respect to reflecting the proportion of caused events in the population as a whole. This is precisely where the dispute lies. Is the sample random in the required sense? One cannot answer this question too confidently in the absence of a detailed examination of the concept of randomness, since our intuitions about randomness (construed epistemologically) aren't too strong. But the randomness of the sample seems highly doubtful in view of the fact that a condition of admission into the sample is having the characteristic in question of being caused. Thus the sample belongs to the class of samples with the highly special characteristic of having only caused events in it; and thus such samples can be representative only if the set of all events is such that practically all or all of them are caused. If not practically all events are caused, then we could be sure in advance that our particular sample comes from a subclass of those samples among which representative ones are very rare. The assertion of randomness seems very doubtful, and therefore the probability assertion also seems doubtful. The doubt is not empirical, note; it is purely logical, like doubt about the validity of a sketchy deductive argument. A number of metaphysical questions may be amenable to this sort of treatment, if we have an appropriate concept of randomness.

E. Decisions.

The application of this epistemological concept of probability to decision problems is perfectly straightforward. Let us take utility to be interval measurable (as is customary), and let us define the expectation of an event $E$, for an agent whose body of knowledge is $K_1$, to be the utility of $E$, multiplied by the probability of $E$, in the following way:

**Definition**  If $\Pr_{K_1}(E, p, q)$, and $U$ is the utility function of the agent, then $ME(E) = (U(E) \cdot p, U(E) \cdot q)$. 

292
In view of the fact that we have intervals here rather than simple numbers, we cannot simply adopt the standard Bayesian maxim: maximize your mathematical expectation in choosing among available acts. We can adopt various versions of it: maximize the minimum expectation \( \sum U(C_i) p_i \); maximize the maximum expectation \( \sum U(C_i) q_i \); maximize the average expectation \( \frac{1}{2} \sum U(C_i) p_i + \frac{1}{2} \sum U(C_i) q_i \); or some weighted average \( r \cdot \sum U(C_i) p_i + s \cdot \sum U(C_i) q_i \). Or we can adopt a minimax principle of some sort, which may lead to different results. Again since we are dealing with pairs of numbers, we have the freedom to consider a wide variety of possible rules. Although clearly a lot of complications are possible, and there is a fruitful field for research here, there are no special or unfamiliar problems that arise in connection with a theory of rational action based on an epistemological concept of probability.

IV

I have argued that we could achieve plausible answers to a number of problems, philosophical and otherwise, if only we had a clear and useful conception of randomness; and that given such a conception all probability statements could be construed—or rather reconstructed—as statements based on (a) knowledge of a relative frequency or measure and (b) knowledge concerning an individual such that relative to that knowledge the individual is a random member of one class with respect to membership in another class. The relation is properly a metalinguistic one, and probability is a metalinguistic concept, so that in fact what one must define are certain relations holding among terms, sentences, etc. Nevertheless, it is far more convenient to talk as if the relations held between objects and classes.
There are those who say that it is obvious that not all probability statements can be based on statistical statements, but the examples they use to support their claim always seem to admit of some sort of statistical reconstruction. This is not merely my own opinion, but also the opinion of those who, unlike me, want to offer an empirical limiting frequency, or an empirical dispositional, analysis of probability statements. They too tend to say: no measure statement, no probability.

I have also argued, much more hesitantly, for a couple of principles which would, in any situation of a fairly simple sort, pick out one class as a reference class—or, if it allowed more than one, would at least ensure that the relevant measure was the same in all the possible reference classes. Perhaps someone will be able to find either some holes in or some more persuasive support for the principles I have enunciated. Or perhaps someone will find some principles that are yet more persuasive. In any event, I take this to be the most fundamental problem in both inductive logic and epistemology. It is worthy of our most serious efforts.